Reply to "Comment on Nonlinear viscosity and Grad's method"

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We show that while Eu's claim is true that we made a mistake regarding the asymptotic behavior of his theory is true, the correct asymptotic behavior cannot have a physical meaning. We analyze in more detail his theory for a dilute gas of rigid spheres, and show that in some cases it predicts a negative value of the xx component of the pressure tensor.

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There are two main objections raised by Eu [1] in the abstract of his Comment on our work [2]. The first objection is that we made a mistake regarding the asymptotic behavior of the xx component of the pressure tensor P_{xx} given by his theory [3,4]. The second one is that if the velocity gradients in the transverse components of the stress are missing, then this implies a vanishing shear viscosity. We will follow Eu's notation closely in order to facilitate the reading of this work. Further, he claimed that many of his results are valid for gases and liquids. Here we will restrict the discussion for dilute gases.

Let us start by analyzing the second point. The Navier-Stokes equations provide a sound theory that can be derived using the kinetic theory of gases [5] or using macroscopic arguments [6]. In the Navier-Stokes regime, the following constitutive relation for dilute gases holds [5]:

$$\mathbf{P}=p\boldsymbol{\delta}-2\eta_0[\nabla\mathbf{u}]^{(2)}, \tag{1}$$

where $\mathbf{P}$ is the pressure tensor, p the hydrostatic pressure, $\boldsymbol{\delta}$ the unit tensor, η_0 is the shear viscosity in the Navier-Stokes regime that is independent of $\nabla\mathbf{u}$, and $[\nabla\mathbf{u}]^{(2)}$ denotes the symmetric traceless tensor formed from the tensor $\nabla\mathbf{u}$. In the case of an unidirectional flow we have,

$$\begin{aligned}\mathbf{u}(x,y,z,t)&=(u_x(x,y,z,t),u_y(x,y,z,t),u_z(x,y,z,t))\\&=(u_x(x,y,z,t),0,0).\end{aligned} \tag{2}$$

Notice that we may consider the case in which $u_x(x,y,z,t)=u_x(x,t)$; as an specific example of this situation we can mention the case of a stationary shock wave, where $u_x(x,y,z,t)=u_x(x)$, which has been extensively studied, see, for example, [7–9] and references therein. If $u_x(x,y,z,t)=u_x(x,t)$, then $[\nabla\mathbf{u}]^{(2)}$ is diagonal and the nonvanishing components are given by

$$[\nabla\mathbf{u}]^{(2)}_{xx}=\frac{2}{3}\frac{\partial u_x}{\partial x},\quad [\nabla\mathbf{u}]^{(2)}_{yy}=-\frac{1}{3}\frac{\partial u_x}{\partial x},$$

$$[\nabla\mathbf{u}]^{(2)}_{zz}=-\frac{1}{3}\frac{\partial u_x}{\partial x}. \tag{3}$$

So, there are specific situations—a traveling shock wave—for which there are no gradients in the transvere directions and the viscosity is not zero, which is in contradiction to the statement made by Eu. We would like to emphasize that so far we have taken the Navier-Stokes regime, but from now on we will consider extensions to the Navier-Stokes equations. We will use the notation $\partial u_x/\partial x\equiv\partial_x u_x$.

There are several statements made by Eu [1], which in our opinion, lack support and therefore the "categorical" way in which they are stated is misleading. Take, for example, the statement, "Besides, because the stress evolution equations of theirs cannot be shown to be consistent with the laws of thermodynamics, their nonlinear viscosity formula cannot be meaningful basis for comparisons with well tested and thermodynamically consistent result as the non-Newtonian viscosity formula given in Eq. (2)." What are the laws of thermodynamics that Eu is referring to? They certainly cannot be the laws of thermodynamics for homogeneous systems [10] since there are inhomogeneities in the system. Also, he cannot be referring to the laws of linear irreversible thermodynamics (LIT) [11], a sound and well accepted theory, since the problem of nonlinear viscosity is in fact beyond the linear regime and therefore outside LIT. The laws of thermodynamics that Eu is talking about must then be an extension of LIT. However, Eu's version of thermodynamics is only a theory among many others [12], and the question of what is the correct thermodynamics beyond LIT is in our opinion an open task [13]. We think that the statement made by Eu is unfair because it discredits our results without a basis. Also, it would be important to know to what specific law of thermodynamics Eu thinks our results are at odds. We will see later that it is true that our results [2] have a more limited scope than we thought, but the reason for this comes from other well sustained objections.

Let us now analyze the point of the asymptotic behavior for P_{xx} when $([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}\rightarrow\infty$, and in order to make the discussion clear we will introduce the following therminology; P^{Eu}_{xx} will denote the xx component of the pressure tensor in Eu's theory [4] for constant temperature [see Eq. (8.67)],

$$P^{Eu}_{xx}-p=-2\eta_0\frac{\sinh^{-1}\kappa_l}{\kappa_l}[\nabla\mathbf{u}]^{(2)}_{xx}, \tag{4}$$

where p, the hydrostatic pressure, and κ_l is defined by [4] [see Eq. (8.66) and below Eq. (8.39)],

$$\kappa_l=(mkT)^{1/4}\frac{\sqrt{\eta_0}}{pd}([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}, \tag{5}$$

where, according to Eu, d is the diameter of the molecule, T the temperature, k the Boltzmann constant, and m the mass of the atoms. By P_{xx}^{Ka} we denote the xx component of the pressure tensor as obtained by Karlin *et al.* [14], which for the Maxwell model reads as

$$P_{xx}^{Ka}=p(1-R(g)g), \tag{6}$$

where $g\equiv a_l^* = \eta_0 \partial_x u_x/p$ and

$$R(g)=\frac{-3-2g+3\sqrt{1+(4/3)g+4g^2}}{4g^2}. \tag{7}$$

P_{xx}^{L} and P_{xx}^{NL} will denote our expressions as given by Eqs. (25) and (45) of our previous work [2], namely,

$$P_{xx}^{L}=p\frac{1+g}{1+7g/3},$$

$$P_{xx}^{NL}=p\left(-\frac{98}{3}a_l^*-13+\frac{2}{3}\sqrt{2401a_l^{*2}+1974a_l^*+441}\right). \tag{8}$$

In our work [2] we mentioned that for a dilute gas of rigid spheres, see Eq. (9) below, P_{xx}^{Eu} vanished in the limit $a_l^*\to -\infty$ $(\partial_x u_x \to -\infty)$, indeed an incorrect statement as Eu [1] points out. It turns out from Eqs. (4) and (5) [1] that $P_{xx}^{Eu} \to \pm\infty$ when $[\nabla \mathbf{u}]_{xx}^{(2)} \to \mp\infty(([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}\to\infty)$.

Let's now analyze Eu's remark, ‘‘Incidentally, before attempting comparisons with other theories, their nonlinear viscosity formula should have been tested against some simulation or experimental results for nonlinear viscosity reported in the literature, as has been done for Eq. (2) since 1983 over a number of occasions [6–13]. Plausible limits of the nonlinear viscosity in special cases are by no means of assurance for its veracity in the face of experiment.'' The previous remark by Eu would imply that since his theory has been claimed to reproduce experimental data, we should accept that the xx component of the pressure tensor can be negative since $P_{xx}\to-\infty$ when $[\nabla\mathbf{u}]_{xx}^{(2)}\to\infty$. We, of course, disagree with this statement since a negative value for P_{xx} is unacceptable even if there is agreement for other quantities with experimental data. Of course there could be agreement with experiment in some range of values of $([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}$ with positive values for P_{xx}, but such an agreement does not imply that for all range of values of $([\nabla\mathbf{u}^{(2)}:[\nabla\mathbf{u}^{(2)})^{1/2}$ the formula should be valid. It is well known that a particular expression in a physical theory may lead to unphysical results in some limits as it happens for Eu's expression for P_{xx}. We again think that Eu's remark is unfair since we find it difficult to accept a theory, which as mentioned by Eu, is consistent with Eu's version of thermodynamics and gives unphysical results, but we agree that plausible limits are by no means assurance of veracity.

In order to go further we will assume that $\mathbf{u}(x,y,z,t)=u_x(x,t)$, so that $([\nabla\mathbf{u}]^{(2)}:[\nabla\mathbf{u}]^{(2)})^{1/2}=\sqrt{2/3}|\partial_x u_x|$. Considering the rigid sphere model and using the Navier-Stokes

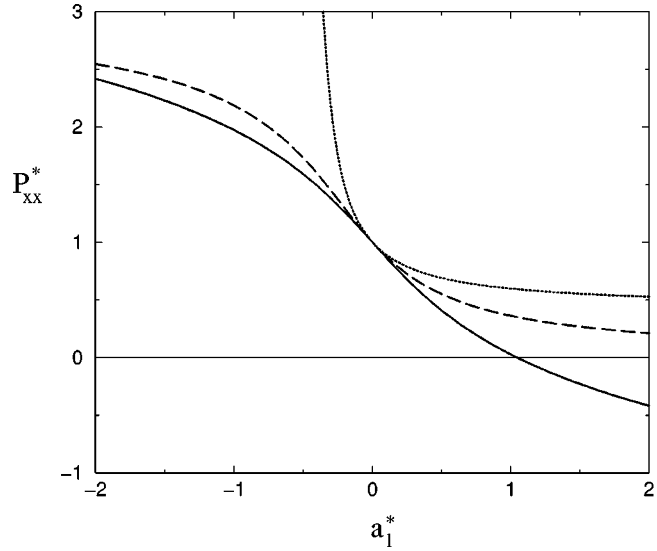

FIG. 1. Reduced xx component of the pressure tensor vs reduced longitudinal rate, P_{xx}^* vs a_l^*. Solid line, P_{xx}^{*Eu} (rigid spheres); long-dashed line, P_{xx}^{*Ka} (Maxwell model); dotted line, P_{xx}^{*NL} (rigid spheres).

result for the viscosity η_0 [5] [see also Eq. (19) in Ref. [2]], we can obtain from Eqs. (3)–(5) the following expression for P_{xx}^{Eu}:

$$P_{xx}^{*Eu}\equiv P_{xx}^{Eu}/p=1-\frac{1}{24}\frac{\sinh^{-1}\sqrt{\frac{32}{15}}\pi^{1/4}|a_l^*|\sqrt{32}\sqrt{15}a_l^*}{\pi^{1/4}|a_l^*|}. \tag{9}$$

In Fig. 1 the reduced xx components of the pressure tensor for its three different forms that we have discussed above are given. Here, reduced means the xx component of the pressure tensor divided by the hydrostatic pressure p. It is important to mention that while Eu and our results correspond to the stationary case, the result by Karlin *et al.* [14] corresponds to a nonstationary situation, but presumably we can take the stationary case in which u_x is time-independent and use Eq. (7) for this case. Assuming that the distribution function does not change when the y component of the molecular velocity is interchanged with its z component, see Eq. (3) in Ref. [2], it follows that $P_{yy}=P_{zz}$. Since the trace of the pressure tensor is equal to three times the hydrostatic pressure—a condition that was misprinted below Eq. (8) of our work [2]—it follows that $P_{yy}=P_{zz}$ and then we obtain the relation, $P_{yy}^*=3/2-P_{xx}^*/2$ $(P_{yy}^*\equiv P_{yy}/p)$. This means that unphysical results are obtained if $P_{xx}^*>3$, a point that will be analyzed further below. In Fig. 1 it is shown that P_{xx}^{*Eu} is negative for a_l^* greater than about 1, so Eu's theory can at most be valid for $a_l^*\in(-\infty,\approx 1]$.

Our expressions for P_{xx}^{L} and P_{xx}^{NL} also have a certain range of validity [2] but the range is smaller than we thought. As Santos [15] pointed out, P_{yy} should also be positive; a fact that we did not analyzed in our previous work. It turns out that P_{xx}^{NL} makes physical sense $(P_{xx}^{NL},\ P_{yy}^{NL}\geqslant 0)$ for $a_l^*\in[-5/14,\infty)$, the limit of P_{xx}^{NL} when $a_l^*\to-\infty$ exists but has no physical meaning,. Similary, P_{xx}^{L} has physical sense for $a_l^* \in [-1/3,\infty)$. At $a_l^* = -1/3$ where P_{xx}^{L} and P_{xx}^{NL} have physical meaning, it turns out that the percentage difference between both expressions is about 15%, which means that the nonlinear contributions of the moments in the collision term are not small in general.

For Eu's theory, one has to analyze the range of longitudinal rates for which P_{yy}^{Eu} is greater or equal than zero, and it turns out that P_{xx}^{*Eu} given by Eq. (9) is greater than 3 for values of a_l^* less than about -5, so the range of validity of eq. (9) is for $a_l^* \in [\approx -5, \approx 1]$. As mentioned by Eu [1], his expression for the nonlinear viscosity is valid under the approximation that $P_{xx} - P_{yy}$ is small, but using Eq. (9) it turns out that percentage difference between P_{xx}^{*Eu} and P_{yy}^{*Eu} can be greater than 10% if a_l^* is outside the interval $[-0.05, 0.05]$. For $a_l^* \in [-0.05, 0.05]$ the percentage difference between P_{xx}^{*Eu} and P_{xx}^{*Ka} is less than about 0.25%, and between P_{xx}^{*Eu} and P_{xx}^{*NL} less than about 1%. This means that for certain ranges of a_l^* in which Eu's formula is expected to be valid, P_{xx}^{*Eu}, P_{xx}^{*Ka}, and P_{xx}^{*NL} basically give the same results.

On the other hand, $P_{xx}^{*Ka} \in [0,3] \forall a_l^* \in \mathbf{R}$, and therefore P_{xx}^{*Ka} does not gives rise to unphysical results. We have the limits $P_{xx}^{Ka}/p \to 0$ as $\partial_x u_x \to \infty$ and $P_{xx}^{Ka}/p \to 3$ as $\partial_x u_x \to -\infty$, a limit that was given incorrectly in Ref. [2].

Finally, we would like to point out that while Eu does not expect two different material functions (viscosities) depending on the sign of the velocity gradient, the results provided by Karlin *et al.* [14], Santos [15], and us [2] are in contrast with Eu's expectation. Rephrasing Eu plausible expectations are by no means assurance of veracity, but of course the experiment or simulations have the last word. We have been unable to find experiments in the references provided by Eu [1] to clarify this issue for the specific case considered here (unidirectional flow of a dilute gas with no heat flux), and for the models discussed in this work, simulations are also apparently lacking. It should be pointed out that many important points of the formulations by Eu [16], Karlin and coworkers [14,17,18], Santos [15], and others [19] were left out for reasons of space, but the reader can resort to the references provided here. It seems that more work is needed to completely clarify the problem about the physical meaning of nonlinear viscosity in some situations, although a great deal of understanding has recently been achieved.

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